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Hopf solitons and elastic rods

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Hopf solitons in the Skyrme-Faddeev model are string-like topological solitons classified by the integer-valued Hopf charge. In this paper we introduce an approximate description of Hopf solitons in terms of elastic rods. The general form of the elastic rod energy is derived from the field theory energy and is found to be an extension of the classical Kirchhoff rod energy. Using a minimal extension of the Kirchhoff energy, it is shown that a simple elastic rod model can reproduce many of the qualitative features of Hopf solitons in the Skyrme-Faddeev model. Features that are captured by the model include the buckling of the charge three solution, the formation of links at charges five and six, and the minimal energy trefoil knot at charge seven.

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The Skyrme-Faddeev model [1] is a field theory in three-dimensional space. It possesses finite energy topological soliton solutions that have a novel string-like structure and are known as Hopf solitons, as the topological classification is via the integer-valued Hopf charge Q. This is a linking number, rather than the more familiar degree that appears in most theories with topological solitons.

It has been suggested [2] that the model may be of relevance for a low energy description of QCD, where the solitonic strings could describe glueballs. Other applications have also been proposed in the context of condensed matter physics, regarding the study of two charged condensates [3].

Substantial numerical work [4–8] has produced a comprehensive catalogue of minimal energy solitons in the Skyrme-Faddeev model. For $Q = 1$ and $Q = 2$ the minimal energy solitons are axially symmetric, but for larger values of Q they are more exotic and include knots and links. These solutions have been obtained using sophisticated, and computationally intensive, numerical simulations of the highly nonlinear field theory. The aim in this paper is to provide a simple approximate description of Hopf solitons in terms of the string core, thereby reducing the three-dimensional field theory to an effective one-dimensional description. The resulting theory is that of a generalized elastic rod, extending that of the classical Kirchhoff rod. It is shown that a minimal extension provides a simple model that is already capable of reproducing many of the qualitative features of Hopf solitons.

The Skyrme-Faddeev field is a map $\boldsymbol{\phi} : \mathbb{R}^3 \mapsto S^2$, which is realized as a three-component unit vector $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$. As this paper is concerned only with static solutions then the model can be defined by its energy

$$E^{\mathrm{SF}} = \int \partial_i\boldsymbol{\phi} \cdot \partial_i\boldsymbol{\phi} + \frac{1}{2}(\partial_i\boldsymbol{\phi} \times \partial_j\boldsymbol{\phi})^2 \; d^3x. \qquad (1)$$

The first term in the energy is that of the usual $O(3)$ sigma model and the second is a Skyrme term, required to provide a balance under scaling and hence allow solitons with a finite non-zero size.

Finite energy boundary conditions require that the field tends to a constant value at spatial infinity, which is chosen to be $\boldsymbol{\phi}(\infty) = (0, 0, 1)$. This boundary condition compactifies space to S^3, so that the field becomes a map $\boldsymbol{\phi} : S^3 \mapsto S^2$. Such maps are classified by $\pi_3(S^2) = \mathbb{Z}$, so there is an integer-valued topological charge Q, the Hopf charge. It has a geometrical interpretation as the linking number of two closed curves obtained as the preimages of any two distinct points on the target two-sphere. A natural definition of the position of the soliton is provided by the preimage curve of the point $\boldsymbol{\phi} = (0, 0, -1)$, which is antipodal to the vacuum value. The position of a Hopf soliton is therefore a closed string, or possibly a collection of closed strings since the preimage of any point may contain disconnected components. This is the novel string-like aspect of Hopf solitons. An energy bound of the form $E \geq a\,Q^{3/4}$, has been proved [9], where a is a constant, and it is known that the fractional power is optimal [10]. The sublinear growth has a simple physical explanation, in that Hopf charge can be accrued by the linking of distinct components of the position string (or by self-linking of a single component), so that the total Hopf charge can be greater than the naive sum of its components.

Generically, the cross-section through the string position locally resembles a planar soliton of the $O(3)$ sigma model. The planar soliton has an internal phase, which may vary along the length of the curve to provide a twist. The simplest axially symmetric Hopf soliton of charge Q may be pictured as a circular string, with a constant twist rate, in which the total change in the phase angle around the circle is $2\pi Q$. Such solutions exist for all Q, but their energy grows linearly with large Q, in contrast to the $Q^{3/4}$ growth of the minimal energy solitons. Only for $Q = 1$ and $Q = 2$ are the minimal energy solitons of this axial form. For $Q > 2$ there is a buckling instability

and in fact the minimal energy soliton with $Q = 3$ has a buckled conformation.

The $Q = 4$ minimal energy soliton is axially symmetric but is anomalous in that the cross-section is described by a double planar soliton, with total phase twist 4π. There is a local minimum of the energy that describes a buckled $Q = 4$ solution but it has an energy that is a few percent above that of the global minimum. There is a further solution at $Q = 4$ that consists of two $Q = 1$ solitons that are linked once. If one links once a ring of charge Q_1 and a ring of charge Q_2, the resulting configuration has total charge $Q = Q_1 + Q_2 + 2$ due to the extra linking of the preimage curves of the components. The charge four link has an energy between that of the buckled solution and the global minimum.

Minimal energy solitons with $Q = 5$ and $Q = 6$ are both links, with two components and a single linking. In the first case the components have charges one and two, and in the second case both components have charge two. The first minimal energy knot appears at $Q = 7$. It takes the form of a trefoil knot, which has crossing number three, and has four units of twist, to make the total Hopf charge equal to seven.

For later comparison, a selection of minimal energy Hopf solitons are displayed in the top row of Figure 1. For clarity the string position is displayed by plotting a tube around the string, given by an isosurface where $\phi_3 = -0.8$. The red curve indicates the twist and is plotted in a similar fashion from the preimage of a point close to the vacuum value.

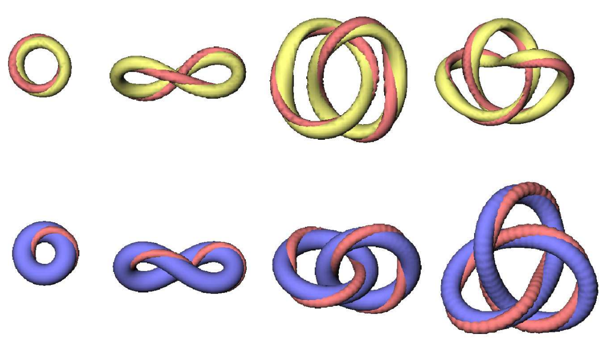

FIG. 1: *Top row:* The string position (yellow tubes) for minimal energy Hopf solitons in the Skyrme-Faddeev model for charges $Q = 1, 3, 6, 7$. The red curves indicate the twist. *Bottom row:* The corresponding minimal energy elastic rods (blue tubes). To aid visualization, the displayed rod thickness is only half the true thickness.

The general form of an elastic rod energy will now be derived from the Skyrme-Faddeev field theory energy (1). The approach taken is to define a field configuration by specifying a closed curve $\boldsymbol{\gamma}(s) \in \mathbb{R}^3$, where $s \in [0, L]$ is an arclength parameter, together with a quasiperiodic twist function $\alpha(s)$ such that $\alpha(L) = \alpha(0) + 2\pi m$, where $m \in \mathbb{Z}$. The field configuration is defined by introducing tubular coordinates centered on the curve and then mapping the tube to the target two-sphere by first twisting each transverse disk using $\alpha(s)$, and then applying a fixed degree one map from the disk to S^2. Physically, the curve may be pictured as the location of the string, the fixed degree one map is the analogue of the planar soliton and the twist function specifies its internal phase.

To make the above construction explicit, let $\kappa(s)$ and $\tau(s)$ denote the curvature and torsion of the curve $\boldsymbol{\gamma}(s)$ and consider the standard Frenet frame with normal $\mathbf{N}(s)$ and binormal $\mathbf{B}(s)$. Take the radius of the transverse disk to be equal to the radius of curvature $1/\kappa(s)$. The relation between the Euclidean coordinates $\mathbf{x} \in \mathbb{R}^3$ and the tubular coordinates s, ρ, θ is

$$\mathbf{x} = \boldsymbol{\gamma}(s) + \frac{\rho}{\kappa}(\mathbf{N}(s)\cos\theta + \mathbf{B}(s)\sin\theta), \tag{2}$$

where ρ and θ are polar coordinates in the unit disk.

The degree one map from the disk to S^2 has the form

$$\boldsymbol{\phi} = (\sin f \cos\theta, \sin f \sin\theta, \cos f), \tag{3}$$

where $f(\rho)$ is a profile function satisfying the boundary conditions $f(0) = \pi$ and $f(1) = 0$.

Assuming the tubular region has no self-intersections, then for points in $\mathbb{R}^3$ inside the tubular region the above construction provides an explicit field $\boldsymbol{\phi}$, and outside this region the field is set to its vacuum value $\boldsymbol{\phi} = (0, 0, 1)$. This defines a field $\boldsymbol{\phi}$ throughout $\mathbb{R}^3$ with a well-defined Hopf charge. In the case that the curve $\boldsymbol{\gamma}$ is an unknot then the Hopf charge is equal to the number of revolutions of the twist function, $Q = m$.

Substituting this ansatz into the Skyrme-Faddeev energy (1) gives, after a lengthy calculation, the energy

$$E = \int_0^L \left\{ b_1 + c_1\kappa^2 \; + \; (\alpha' - \tau)^2 \left(\frac{b_2}{\kappa^2} + c_2 \right) \right.$$
$$\left. + \; \frac{\kappa'^2}{\kappa^4} \left(b_3 + c_3\kappa^2 \right) \right\} ds, \tag{4}$$

where the constants are given in terms of integrals of the profile function and its derivative. The terms in the energy associated with the coefficients b_1, b_2, b_3 derive from the sigma model term and the terms associated with the coefficients c_1, c_2, c_3 are obtained from the Skyrme-term.

The energy (4) is a function of the position curve $\boldsymbol{\gamma}(s)$ and the twist function $\alpha(s)$, and is a natural generalization of the classical Kirchhoff energy for elastic rods. The Kirchhoff energy is recovered if all coefficients except c_1 and c_2 are set to zero. In this case, the two terms correspond to a bending and a twisting energy and the rod length L needs to be fixed.

The main aim of the present paper is to show that an elastic rod model can reproduce many of the qualitative features of Hopf solitons. This motivates a limited

phenomenological approach, in which a simplified form of the energy (4) is considered, by setting some of the coefficients to zero. The remaining coefficients are then fixed by fitting to selected properties of Hopf solitons.

The starting point is the Kirchhoff energy, therefore c_1 and c_2 are taken to be non-zero. To balance the scaling properties of these two terms requires at least one contribution that is obtained from the sigma model term. We choose the simplest possibility, namely to take b_1 to be non-zero, so that there is a contribution to the rod energy that simply involves its length L. The reduced elastic rod energy is therefore defined by setting $b_2 = b_3 = c_3 = 0$. In particular, this means that terms involving the derivative of the curvature are ignored in the reduced theory.

The reduced elastic rod energy contains only three parameters. Two of these parameters simply define the energy and length units of the model, which can be fixed by matching to the energy and length of the $Q = 1$ Hopf soliton. This means that, in suitable units, two of the three coefficients may be set to unity and the energy of the reduced elastic rod model can be taken to be

$$E = \int_0^L \left(1 + \kappa^2 + C(\alpha' - \tau)^2\right) ds. \quad (5)$$

A rod that is a minimizer of the energy (5) is in fact a Kirchhoff rod, as it is also a minimizer of the Kirchhoff energy. The additional feature induced by the first term in (5) is that the rod length L is not fixed, but is itself determined by energy minimization.

Axially symmetric Hopf solitons of charge Q are modelled by circular rods with a linear twist function $\alpha = 2\pi Qs/L$. They have energies and lengths given by

$$E_Q^{\mathrm{O}} = 4\pi\sqrt{1 + CQ^2}, \qquad L_Q^{\mathrm{O}} = 2\pi\sqrt{1 + CQ^2}. \quad (6)$$

Note that this energy formula captures the linear growth for large Q found for axially symmetric Hopf solitons.

As shown by Michell [11] in 1889 , a circular rod will buckle if the total twist $2\pi Q$ exceeds a critical value $2\pi Q > 2\pi\sqrt{3}/C$. For the first buckling instability of an axial rod to occur at $Q = 3$ requires that $\frac{1}{\sqrt{3}} < C < \frac{\sqrt{3}}{2}$. The analysis below reveals that the rod energies underestimate the soliton energies, so twisting is not sufficiently penalized. To minimize this deficiency C should be taken to be close to its upper limit, therefore in the following we set $C = 0.85$. With this value of C the above formulae yield $E_1^{\mathrm{O}} = 17.09$, $L_1^{\mathrm{O}} = 8.55$, $E_2^{\mathrm{O}} = 26.36$, $L_2^{\mathrm{O}} = 13.18$, $E_3^{\mathrm{O}} = 36.96$, $L_3^{\mathrm{O}} = 18.48$. A comparison of the $Q = 1$ and $Q = 2$ rods reveals that $E_2^{\mathrm{O}}/E_1^{\mathrm{O}} = L_2^{\mathrm{O}}/L_1^{\mathrm{O}} = 1.54$. For Hopf solitons these ratios are $E_2^{\mathrm{SF}}/E_1^{\mathrm{SF}} = 1.63$ and $L_2^{\mathrm{SF}}/L_1^{\mathrm{SF}} = 1.45$, so the rod model is reasonably accurate, with an error of around 6% for both the energy and length of the $Q = 2$ solution. The $Q = 1$ axial rod is displayed as the first image in the bottom row of Figure 1.

To avoid self-intersection of the rod we impose an additional constraint by assigning a radius, R, to the cross-section of the rod and demand that this thickened rod does not self-intersect. For the rod thickness to have no influence on the axial solutions requires that $R \leq L_1/(2\pi) = 1.36$. To maximize rod energies, we set R equal to this upper limit.

To numerically compute minimal energy rods the curve $\boldsymbol{\gamma}$ is discretized into a polygonal curve with 100 vertices. A simulated annealing method is used to minimise a discrete version of the energy (5), with a condition of equal edge lengths imposed using a penalty function. A discrete version of curvature is calculated from the angle between two neighbouring edges [12] and the self-intersection of the rod is excluded by imposing an upper limit on the curvature $\kappa \leq 1/R$, and a lower limit on the separation between specific subsets of pairs of vertices, using the algorithm presented in [12]. The twist contribution to the energy is evaluated by identifying the combination $\alpha' - \tau$ with the rate of change of the angle between the material frame of the rod and the twist free Bishop frame, as described in [13].

Our numerical computations determine the energies of the first two buckled rods to be $E_3^{\infty} = 35.35$, and $E_4^{\infty} = 44.16$. The buckled minimal energy $Q = 3$ rod is displayed as the second image in the bottom row of Figure 1, where it can be seen that it closely resembles the corresponding Hopf soliton.

Links can be formed by putting together two rods, provided their radii are first increased to allow a sufficient gap to accommodate the other component of the link. Let $E_Q^{\odot}$ denote the energy of the axial rod with a radius at least $2R$, so that there is enough room to slip another rod through its centre. Note $L_3^{\mathrm{O}} > 2L_1^{\mathrm{O}}$, hence for $Q \geq 3$ the axial rod is already large enough to insert another rod giving $E_Q^{\odot} = E_Q^{\mathrm{O}}$. However, for $Q = 1$ and $Q = 2$ the natural radius must be increased to give

$$E_Q^{\odot} = \frac{\pi}{R}(4R^2 + 1 + CQ^2). \quad (7)$$

With the values of C and R given earlier this formula results in the energies $E_1^{\odot} = 21.36$, $E_2^{\odot} = 27.25$. Forming a single link between a rod of charge Q_1 and a rod of charge Q_2 produces a link of charge $Q = Q_1 + Q_2 + 2$. We denote the energy of such a linked rod configuration by $E_Q^{Q_1,Q_2}$. If both Q_1 and Q_2 are less than three then both components of the link will be axial rods and the energy is simply obtained by addition as $E_Q^{Q_1,Q_2} = E_{Q_1}^{\odot} + E_{Q_2}^{\odot}$. This gives the link energies $E_4^{1,1} = 2E_1^{\odot} = 42.73$, $E_5^{2,1} = E_2^{\odot} + E_1^{\odot} = 48.62$, $E_6^{2,2} = 2E_2^{\odot} = 54.51$. Thus for $Q = 4$ the link is slightly lower in energy than the buckled rod, which agrees with the result for Hopf solitons, where solutions of both types also exist. Recall that the minimal energy Hopf soliton for $Q = 4$ is anomalous, in that its cross-section is a double planar soliton, therefore in its current form the elastic rod model is unable to describe

this soliton. It is possible to generalize the rod model to a cross-section with arbitrary degree, but this refinement is more appropriate for future investigations that include all the possible terms in the energy. A lower bound on the energy of a link formed from rods of charge three and two is $E_7^{3,2} \geq E_3^{\infty} + E_2^{\odot} = 62.60$. The energies of trefoil knots with charge Q are denoted by E_Q^K, and have $Q-3$ turns, with a contribution of three to the Hopf charge due to the self-linking of the trefoil knot. Our numerical computations provide the following trefoil knot energies $E_5^K = 53.09,\ E_6^K = 57.20,\ E_7^K = 61.32$.

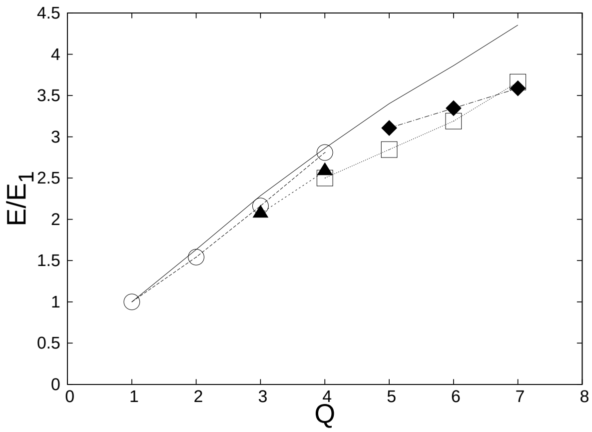

FIG. 2: The data points are (normalized) elastic rod energies: circles for circular rods, triangles for buckled rods, squares for links, and diamonds for knots. The upper solid curve is the (normalized) minimal energy of Hopf solitons in the Skyrme-Faddeev model.

Comparing the above link and knot energies shows that links are preferred for $Q=5$ and $Q=6$, but at $Q=7$ the knot energy is below the lower bound of the link. These results show that the forms of minimal energy rods for $Q=5,6,7$ are in agreement with those of minimal energy Hopf solitons. The final two plots displayed in the bottom row of Figure 1 are the minimal energy $Q=6$ and $Q=7$ rods, whose forms compare reasonably well with the corresponding minimal energy Hopf solitons presented in the top row.

The data points in Figure 2 are the various rod energies, normalized by the energy E_1^{O} of the $Q=1$ rod. For comparison, the upper solid curve is the minimal energy in the Skyrme-Faddeev model, normalized by the energy E_1^{SF} of the $Q=1$ Hopf soliton. Although the rod energies grow too slowly in comparison to the soliton energies, we have seen that several qualitative features are correctly reproduced regarding the minimal energy configurations, that is, axial for $Q=1,2$, buckled for $Q=3$, links for $Q=5,6$ and the first minimal knot at $Q=7$.

It is perhaps not surprising that the rod energies underestimate the Hopf soliton energies, because the non-local interaction between different parts of the rod is very simplistic and does not provide a direct contribution to the energy. It might be possible to produce a more accurate model by introducing a more sophisticated non-local interaction, but then some of the simplicity and elegance of the current model would be lost. Similarly, the extra terms neglected to produce the reduced rod model (for example, terms involving the derivative of the curvature) could also be considered and will again lead to increases in rod energies. This may provide a more accurate quantitative rod model, and could be investigated in the future.

It is certainly of interest to obtain an improved quantitative description of Hopf solitons, to build on the successful qualitative results presented here. An accurate rod approximation would be a very useful tool for investigating Hopf soliton issues that are currently difficult to study within the field theory, such as the existence of non-torus knots and the structure of solitons for large Hopf charges.

Finally, note that if the rod energy simply consisted of the first term in (5) then our problem would coincide with the construction of ideal knots and links [14], in which the energy function is the length of the rod for a fixed thickness. The energy (5) is therefore an interesting hybrid of the Kirchhoff elastic rod and ideal knot energies.

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